Reply by Author to A. N. Tifford

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THE paper published by the author[7] in the February issue of AIAA J. is part of the results of a long range research program on rheology conducted by the author. It has two purposes, namely, to discuss the possible similarity solutions and to apply the group-theoretic method to such an analysis.

The selection of an infinite flat plate as a problem and the importance of the power law model in non-Newtonian flow need no defense. The fact that some authors[11,12] treat infinite and semi-infinite flat plate problems together does not mean the infinite flat plate itself does not constitute a research problem. The importance of the power law model of the Ostwald-de Waele model can be shown by simply counting the number of publications in recent years on this model. Although it is empirical, it has been observed that a large number of non-Newtonian fluids behave according to this simple law. The discusser's so-called "more general laws of viscosity," Eqs. (1, 2 and 3) in the Comment, is nothing more than one of an infinite number of possible mathematical functions. Any discussion on them will be against the "Plea, and Clarion Call" of the Editor.[10]

The group-theoretic method used in the original paper[7] is a method based on the concepts developed from the theory of transformation groups. It was first given by Birkhoff[1] and then by Morgan[6] and is discussed in detail in a recent book by Hansen.[2] Kline's new book[4] has also an excellent discussion on it. The author would like to point out that the method used in the paper by the discusser (Ref. 9 in his Comment) is not the group-theoretic method as he claims.

The author is happy to know that Wells[12] obtained the same results by the usual free-parameter method. This result came to the author's attention after the paper was published. Even so, the author does not regret publishing this paper. Checking the same results by entirely different methods is also worth doing. Two recent works on the group-theoretic method are of this nature. One of them is a recent report (1963) by Manohar,[5] of the Mathematics Research Center of the University of Wisconsin, in which the reresults of Schuh (1955) on the unsteady boundary-layer flow[9] and those of Hansen (1958) on three-dimensional boundary-layer flow[3] are checked by the group-theoretic method. The results in the original works were obtained by the free-parameter method. The other is the recent work (1965) by Rao[8] in which he was trying to justify the form of the well-known solution of von Karman's problem of a rotating disk (1921) by the method of group theory. The contribution made in these two reports to our understanding of similarity solutions is the same as that in the original works.

References

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[9] Schuh, H., "Uber die ähnlichen Lösungen der instationären laminaren Grenzschichtgleichungen in inkompressiblen Strömungen," *50 Jahre Grenzschichtforschung* (Friedrich Viewig & Sohn, Braunschweig, Germany 1955), p. 147.

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Comment on "Perturbation Solutions for Low-Thrust Rocket Trajectories"

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THE solutions presented by D. P. Johnson and L. W. Stumpf (Ref. 1) to a specific form of the problem of low-thrust trajectories, namely trajectories arising from the application of low thrusts at constant angle to the radius vector, suffer, as the authors themselves recognize, from the disadvantage of being truncated series solutions with undefined convergence properties. Thus, in the specific example treated, even the second-order theory yields results whose accuracy is doubtful and a priori unascertainable beyond the first revolution of the trajectory.

The method described in Refs. 2–4 can in fact quite simply be extended to solve, to a very good approximation, the problem, treated in Ref. 1, generalised to remove the limitations of constant satellite mass and initially circular parking orbit and extended to cover a considerably larger stretch of the trajectory.

Thus, referring to Eqs. (3) of Ref. 4, these, in the circumstances described in Ref. 1, become

$$[(d^2u)/(d\theta^2)] + u = (1/p^2) - \{[(\cos(\psi + \beta)]/[(A - \tau)u^2p^2 \cos\beta]\} \quad (1)$$

$$(dp^2)/(d\theta) = (2 \sin\psi)/[(A - \tau)u^3]$$

where β is the elevation of the trajectory, and ψ is the constant angle of the thrust vector to the radius vector. If the elevation β is small, an approximate form of the first equation in (1) is

$$[(d^2u)/(d\theta^2)] + u = (1/p^2) - \{\cos\psi/[(A - \tau)u^2p^2]\} \quad (2)$$

We note that, if ψ is little different from $\pi/2$, the error involved in using $\cos\psi/(A - \tau)$ for $[\cos(\psi + \beta)]/[(A - \tau)\cos\beta]$ on the right-hand side of (2) is of order higher than $1/A$ and, hence, need not affect significantly any general solution derived in that range of ψ. If, on the other hand, ψ is significantly different from $\pi/2$, then this substitution is fully justified provided, as is assumed, β is small throughout the portion of the trajectory considered. Thus, such trajectories must originate from parking orbits of small eccentricities. The second equation in (1) is exact with ψ = const and the

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